Decision Rules Revealing Commonly Known Events*

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Abstract

We provide a sufficient condition under which an uninformed principal can infer any information that is common knowledge among two experts, regardless of the structure of the parties' beliefs. The condition requires that the bias of each expert is less than the radius of the smallest ball containing the action space.

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1 Introduction

In this note, we ask under what conditions an uninformed principal can infer any information that is common knowledge among two experts. We show that there exists a decision rule which makes it optimal for the experts to reveal the commonly known information if the supremum of each expert's bias relative to the principal's optimal action is less than the radius of the smallest ball that contains the action space.[1] This condition becomes necessary if the principal does not know the structure of the experts' biases and is concerned with the worst-case scenario.

The rule constructed in this note is simple and depends only on the boundaries of the action space, but is independent of other details of the environment, such as beliefs, directions of the biases, and the relationship between the biases and the experts' information. It implements an optimal action for the decision maker given the commonly known event by the experts if their reports about this event agree and implements a *constant* stochastic action that is independent of the nature of disagreement otherwise. If the experts commonly know the optimal action for the principal, this rule is first best and implements the optimal action in each state.

A related literature on cheap talk with two experts who commonly know the principal's optimal action has focused on establishing conditions under which the decision maker can achieve the *first best* (Krishna and Morgan (2001), Battaglini (2002), Ambrus and Takahashi (2008)). In cheap talk, the decision maker has no commitment power and must take an action that is sequentially rational given her beliefs. In this note, there is no such restriction and, consequently, the condition is much weaker. The key difference is that our condition bounds each expert's bias independently, whereas this is not so in cheap talk.[2]

In this note, incentives for the experts are provided by punishment of disagreements by a stochastic action. This feature also appears in Mylovanov and Zapechelnyuk (forthcoming) and Zapechelnyuk (2012). Mylovanov and Zapechelnyuk's (forthcoming) model is limited to a unidimensional action space and assumes large and opposing biases, where the first best implementation is generally impossible. Zapechelnyuk (2012)

[1]Decision rules in environments with one expert have been studied since Holmström (1977, 1984). For recent work, see, e.g., Koessler and Martimort (2012) and Frankel (2011, 2012) and the references therein.

[2]For instance, if the state space is unidimensional, Proposition 1 in Battaglini (2002) establishes that a necessary and sufficient condition for a fully revealing cheap talk equilibrium is that the sum of the absolute values of the experts' biases is less than half of the measure of the action space. In the environment here, the first best is implementable if and only if each of the expert's biases, rather than their sum, is bounded by that value.

allows multiple experts to collude and derives a condition for the first best; this condition is generally violated in the environments in this note.

2 The Model

There are two experts, $i = 1, 2$, and a principal. The set of actions available to the principal is Y, a compact subset of $\mathbb{R}^{\mathbf{d}}$, $\mathbf{d} \geq 1$. The principal's optimal action x belongs to a set of states $X \subseteq Y$. Each expert $i = 1, 2$ is endowed with an information partition $\mathcal{P}_i$ of X and knows that x belongs to $X_i \in \mathcal{P}_i$. Denote by $\hat{X}$ the smallest subset of X that is common knowledge for the experts.[3]

Let $||\cdot||$ denote a norm on $\mathbb{R}^{\mathbf{d}}$. Let r_Y be the radius of the smallest ball that contains Y and without loss of generality set the center of this ball to be at the origin.

Expert i's most preferred action at state x is given by $x + b_i(x)$, where $b_i(x)$ is i's bias.[4] Each expert minimizes his loss function given by a convex transformation of the squared distance between action $y \in Y$ and i's most preferred action:

$$L_i(x, y) = h_i \left(\|y - (x + b_i(x))\|^2 \right), \ i = 1, 2,$$

where $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ is a weakly convex function with $h_i(0) = 0$; in particular, h_i is non-decreasing.

Let $\Delta(Y)$ denote the set of probability measures on Y (*stochastic actions*). Conventionally, we extend the definition of L_i to $X \times \Delta(Y)$ by $L_i(x, \lambda) = \int_Y L_i(x, y)\lambda(dy)$, $x \in X$, $\lambda \in \Delta(Y)$. A *decision rule* is a measurable function

$$\mu : \quad \mathcal{X}^2 \to \Delta(Y), \quad (\hat{X}_1, \hat{X}_2) \mapsto \mu(\hat{X}_1, \hat{X}_2),$$

where $\mathcal{X} = \mathcal{P}_1 \wedge \mathcal{P}_2$ is the set of subsets of X that can be commonly known by the experts and $\mu(\hat{X}_1, \hat{X}_2)$ is a stochastic action that is contingent on the experts' reports $(\hat{X}_1, \hat{X}_2) \in \mathcal{X}^2$. A decision rule induces a *game*, in which the experts simultaneously make reports $\hat{X}_1, \hat{X}_2 \in \mathcal{X}$ and the outcome $\mu(\hat{X}_1, \hat{X}_2)$ is implemented.

Decision rule μ is *common-knowledge-revealing* if for all $\hat{X} \in \mathcal{X}$,

(C_1) $\mu(\hat{X}, \hat{X})$ is a stochastic action with support on $\hat{X}$, and

(C_2) $L_i(x, y) \leq L_i(x, \mu(\hat{X}_1, \hat{X}_2))$ for all $x, y \in \hat{X}$ and all $\hat{X}_1, \hat{X}_2 \in \mathcal{X}$ with $\hat{X}_1 \neq \hat{X}_2$.

[3]That is, $\hat{X}$ is the element of the meet $\mathcal{P}_1 \wedge \mathcal{P}_2$ that contains x.

[4]One may also assume that $x + b_i(x) \in Y$; our results hold without this assumption.

Condition (C_1) requires that if the experts' reports agree, the decision rule implements an action in the reported set, while condition (C_2) requires that each expert prefers any action in the commonly known set to any action that can possibly be implemented if the experts' reports disagree. These conditions imply that truthtelling is ex-post equilibrium in the common-knowledge-revealing rule if the experts know the rule. Furthermore, truthtelling is optimal even if the experts do not know the rule but know that it will not implement actions outside of the commonly known set.

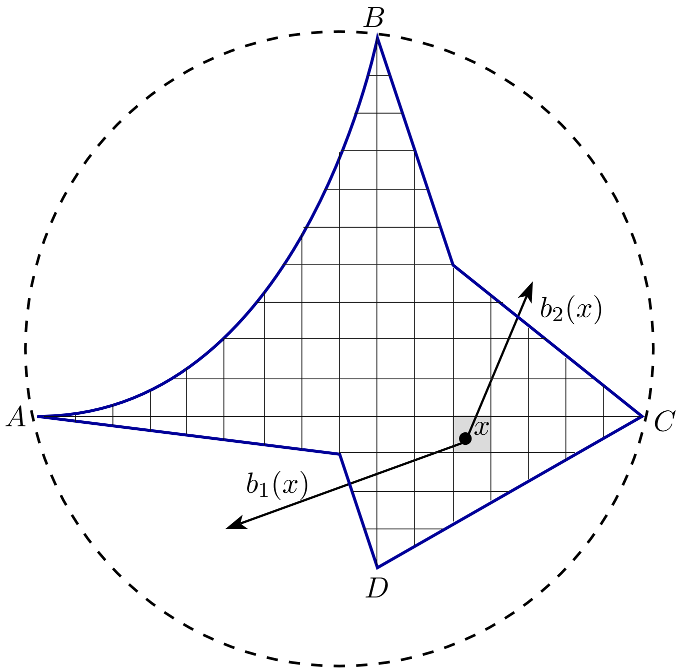

Fig. 1.

Our objective is to construct a common-knowledge-revealing rule. As an example, consider action set Y illustrated by Fig. 1, with the grid representing the experts' common knowledge partition $\mathcal{X}$ (we assume $X = Y$). If state x is as shown on Fig. 1, then it is common knowledge for the experts that x is in the shaded rectangle, $\hat{X}$. In this environment, *how can the principal motivate the experts to reveal their common knowledge at every possible state?*

Consider decision rule μ^* that satisfies (C_1) for all $\hat{X} \in \mathcal{X}$ and $\mu^*(\hat{X}_1, \hat{X}_2) = \lambda^*$ whenever $\hat{X}_1 \neq \hat{X}_2$, where stochastic action λ^* is a solution of:

$$\max_{\lambda \in \Delta(Y)} \int_Y ||y||^2 \lambda(dy), \quad \text{s.t.} \int_Y y\lambda(dy) = 0. \tag{1}$$

That is, any disagreement between the experts results into a stochastic action that has the maximum variance among all random variables on Y with zero expectation.

To construct λ^* geometrically, one draws the smallest ball that contains Y and assigns positive weights on extreme actions in Y that lie on the boundary of this ball, such that the weighted average of those actions is at the center of the ball. So, for the example on Fig. 1, one will assign positive weights on vertices A, B, and C (note that vertex D is strictly inside the smallest ball, so it is assigned zero weight).

Theorem 1 *Decision rule μ^* is common-knowledge-revealing if $||b_i(x)|| \leq r_Y$ for all $x \in X$ and $i = 1, 2$.*

Proof. We need to prove (C_2) for all $\hat{X} \in \mathcal{X}$. For all $x \in X$, all $y \in Y$, and each $i = 1, 2$, by monotonicity of h_i we have

$$L_i(x, y) = h_i\left(||y - (x + b_i(x))||^2\right) \leq h_i\left(||y||^2 + ||x + b_i(x)||^2\right).$$

Next, by assumption, the smallest ball that contains Y is centered at the origin and has radius r_Y. Consequently, $L_i(x, y) \leq h_i\left(r_Y^2 + ||x + b_i(x)||^2\right)$. Also, $\int_Y ||z||^2 \lambda^*(dz) = r_Y^2$, since λ^* that solves (1) must assign positive mass only on subsets of Y that have distance r_Y from the origin (in the example depicted on Fig. 1, λ^* will assign positive mass only on vertices A and B and C). Also, by (1), $\int_Y z\lambda^*(dz) = 0$. Using the above and Jensen's inequality, we obtain for all $x \in X$, all $y \in Y$, and each $i = 1, 2$

$$\begin{aligned} L_i(x, \lambda^*) &= \int_Y h_i\left(||z - (x + b_i(x))||^2\right) \lambda^*(dz) \geq h_i\left(\int_Y ||z - (x + b_i(x))||^2 \lambda^*(dz)\right) \\ &= h_i\left(\int_Y \left\{||z||^2 - 2z \cdot (x + b_i(x)) + ||x + b_i(x)||^2\right\} \lambda^*(dz)\right) \\ &= h_i\left(\int_Y ||z||^2 \lambda^*(dz) - 2(x + b_i(x)) \cdot \int_Y z\lambda^*(dz) + ||x + b_i(x)||^2\right) \\ &= h_i\left(r_Y^2 + ||x + b_i(x)||^2\right) \geq L_i(x, y), \end{aligned}$$

and (C_2) follows immediately. ∎

Note that common-knowledge-revealing rule μ^* is robust to details of the preferences and the information structure of the experts. The only information relevant for construction of μ^* is the principal's set of actions Y and the upper bounds on the experts' biases. The stochastic punishment action λ^* is constant with respect to state space $X \subset Y$, information structure $\mathcal{P}_1$ and $\mathcal{P}_2$ of the experts, as well as parameters of their preferences, h_i and b_i, $i = 1, 2$; to verify the sufficient condition in Theorem 1 one needs to know only the radius of Y and the upper bounds on the experts' biases.

If the principal does not know the biases of the experts, but nevertheless would like the ensure that a common-knowledge-revealing rule exists, the condition in Theorem 1 becomes necessary.

Theorem 2 *Assume that the experts' biases are constant. Then, there exists a pair of directions of biases such that a common-knowledge-revealing rule exists if and only if* $||b_i|| \leq r_Y$ *for each* $i = 1, 2$.

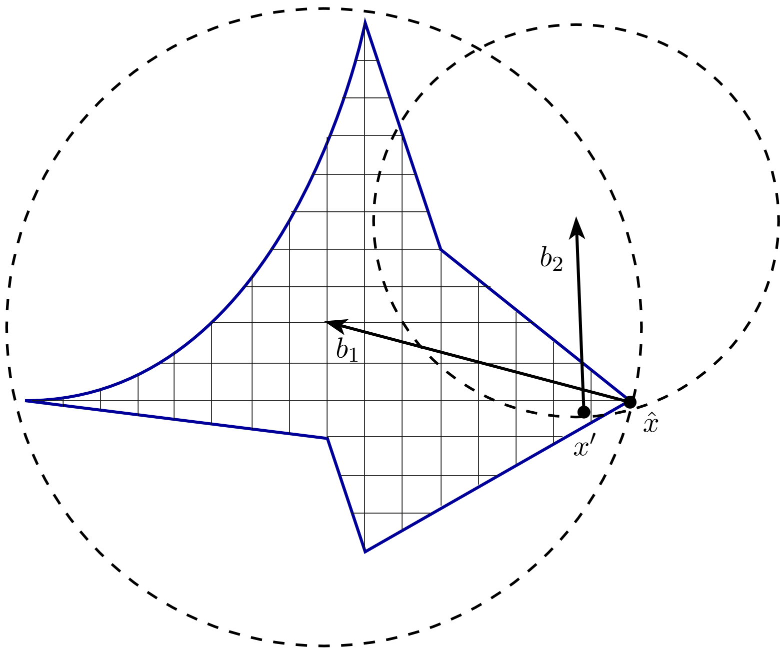

Fig. 2.

Proof. We show that for any action set Y, we can find directions of the biases such that if $||b_i|| > r_Y$ for some $i = 1, 2$, then a common-knowledge-revealing rule does not exist. As illustrated by Fig. 2, consider action $\hat{x}$ in the support of stochastic action λ^* that satisfies (1) and assume that b_1 satisfies $\frac{b_1}{||b_1||} = -\hat{x}$ and $||b_1|| > r_X$. Observe that, since $||b_1|| > r_X$, the set of actions that expert 1 prefers to action $y = \hat{x}$ is inside the larger dashed circle, so she strictly prefers every action in $Y \backslash \{\hat{x}\}$ to $\hat{x}$. Hence, to provide the incentive for 1 to report the element $\hat{X}$ of $\mathcal{X}$ that is commonly known by the experts in $\hat{x}$ truthfully, one must choose $\mu(\hat{X}_1, \hat{X}) = \hat{x}$ for all $\hat{X}_1 \in \mathcal{X}$. If $\hat{X}$ is not a singleton, (C_2) is violated for any $x \in \hat{X}$ and $x \neq \hat{x}$. Let now $\hat{X} = \{\hat{x}\}$. Then, however, in any state expert 2 can report $\hat{X}_2 = \hat{X}$ and obtain action $y = \hat{x}$. Provided the directions of the biases are different enough, there exists $x' \neq \hat{x}$ such that expert 2 (whose set of actions preferred to x' is depicted by the smaller dashed circle) strictly prefers $\hat{x}$ to x'. Consequently, (C_2) is violated for expert 2 at x'. ■

Note that if the experts' information partitions are not identical, there may exist decision rules that extract more information from the experts than their common knowledge.[5] Hence, given the principal's preferences and beliefs about x, the common-knowledge-revealing rule imposes the lower bound on the principal's expected payoff that can be possibly achieved.

References

Ambrus, Attila and Satoru Takahashi, "The multi-sender cheap talk with restricted state spaces," *Theoretical Economics*, 2008, *3*, 1–27.

___ **and Shih-En Lu**, "Robust fully revealing equilibria in multi-sender cheap talk," *mimeo*, 2010.

Austen-Smith, David, "Interested Experts and Policy Advice: Multiple Referrals under Open Rule," *Games and Economic Behavior*, January 1993, *5* (1), 3–43.

Battaglini, Marco, "Multiple Referrals and Multidimensional Cheap Talk," *Econometrica*, 2002, *70*, 1379–1401.

___ , "Policy Advice with Imperfectly Informed Experts," *Advances in Theoretical Economics*, 2004, *4*. Article 1.

Frankel, Alexander, "Aligned delegation," *mimeo*, 2011.

___ , "Delegating multiple decisions," *mimeo*, 2012.

Holmström, Bengt, "On Incentives and Control in Organizations," PhD thesis, Stanford University 1977.

___ , "On the Theory of Delegation," in M. Boyer and R. E. Kihlstrom, eds., *Bayesian Models in Economic Theory*, North-Holland, 1984, pp. 115–41.

Koessler, Frédéric and David Martimort, "Optimal Delegation with multidimensional decisions," *mimeo*, 2012.

Krishna, Vijay and John Morgan, "A Model of Expertise," *Quarterly Journal of Economics*, 2001, *116* (2), 747–75.

[5] For the models in which the experts have differential information, see Austen-Smith (1993), Wolinsky (2002), Battaglini (2004), Levy and Razin (2007), Martimort and Semenov (2008), and Ambrus and Lu (2010).

Levy, Gilat and Ronny Razin, "On the limits of communication in multidimensional cheap talk: a comment," *Econometrica*, 2007, *75*, 885–893.

Martimort, David and Aggey Semenov, "The informational effects of competition and collusion in legislative politics," *Journal of Public Economics*, 2008, *92* (7), 1541–1563.

Mylovanov, Tymofiy and Andriy Zapechelnyuk, "Decision rules for experts with opposing interests," *mimeo*, 2010.

____ **and** ____ , "Optimal arbitration," *International Economic Review*, forthcoming.

Wolinsky, Asher, "Eliciting information from multiple experts," *Games and Economic Behavior*, 2002, *41* (1), 141–160.

Zapechelnyuk, Andriy, "Eliciting information from a committee," *mimeo*, 2012.