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Strategic complements and substitutes, and potential games

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Abstract

We show that games of strategic complements, or substitutes, with aggregation are "pseudo-potential" games. The upshot is that they possess Nash equilibria in pure strategies (NE), even if the strategy sets are not convex; and that various dynamic processes converge to NE. In particular, NE exist in Cournot oligopoly with indivisibilities in production.

Our notion of aggregation is quite general and enables us to take a unified view of several disparate models.

JEL classification: C72

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1. Introduction

Economic theory is replete with examples of what have come to be called, after Bulow et al. (1985), games of strategic substitutes (STS) or strategic complements (STC). They cover phenomena ranging from oligopolistic competition between firms, to the problem of the commons (Dasgupta and Heal, 1979), to macroeconomic coordination failures (Diamond, 1982), to new technology adoption (Katz and Shapiro, 1986), to bank runs (Diamond and Dybvig, 1983). The basic (and most widely espoused—see, e.g., Tirole, 1988) notion of STS/C is founded on cardinal utilities and runs as follows. Assume that players' strategies are totally ordered by their "aggression levels," and w.l.o.g. embedded in the real line. Then the increment in every player's payoff, when he unilaterally deviates from any strategy of his to a more aggressive one, always falls (for STS) or rises (for STC) with increases in his competitors' strategies.

An immediate upshot is that, when his competitors turn more aggressive, the optimal reaction of each player is to become less so (for STS) or more so (for STC).[1] This suggests a broader, and purely ordinal, view of STS/C. Games of *weak* strategic substitutes (WSTS) or complements (WSTC) are those in which there exists a selection from the best reply correspondence of each player, which is nonincreasing (for WSTS), or nondecreasing (for WSTC).[2]

In this paper we focus attention on WSTS/C games which have one further property: the payoff of a player depends only upon his own strategy, and some kind of "market aggregate" of others' strategies. This property is quite common to many examples, including that most famous of all WSTS games: Cournot oligopoly. In the Cournot setting, and indeed in many others, it suffices to take the aggregate to be additive, i.e., just the sum of everyone's actions. However there are situations when the strategic interaction between players is more complex, and simple additive aggregation will not do the job. A broader concept of aggregation is called for, in order to bring out the hidden structure of the games and to render them amenable to our analysis. We motivate and develop such a concept in Section 5.

Once we have aggregation, a very striking thing occurs: in WSTS/C games, players can be thought of as maximizing one *common* payoff function—the "pseudo"-potential—in order to deviate to a best reply (see Theorem 1). This is a variation on the more stringent notion of "ordinal" potential, set forth in Monderer and Shapley (1996), wherein *all* unilateral deviations had to be rank-ordered by the potential. We stress that the pseudo-potential is considerably less faithful to the game than the ordinal potential, or even the "best-reply" potential[3] of Voorneveld (2000).[4] Its maximization yields only a subset of the best replies, and in general it may rank unilateral deviations differently from their concomitant gains

[1] See, e.g., Theorem 4 of Milgrom and Shannon (1994).

[2] While the terms WSTS/C are possibly new (indeed they were suggested by an anonymous referee), such games have already been looked at for some time (see, e.g., Kukushkin, 1994).

[3] But the pseudo-potential reduces to the best-reply potential in the event that all best replies happen to be unique in the game.

[4] See also Huang (2002) and Morris and Ui (2004).

in payoff. Nevertheless the pseudo-potential turns out to be an effective technical tool, enabling us to take a unified and simple view of the dual classes of WSTS and WSTC games.

Up until now the analyses of WSTS and WSTC games have tended to be along quite different lines. For instance, the existence of Nash equilibrium (NE)[5] in WSTC games follows immediately from the Tarski's fixed point theorem, applied to the product of players' best reply selections. This was noticed by Milgrom and Roberts (1990) and Vives (1990), who investigated a subclass of WSTC games known as supermodular.[6] But in WSTS games, this product constitutes a nonincreasing function, and Tarski's theorem is no longer directly applicable (except, of course, in the case of two players, where—reversing the order of one player's strategies—a WSTS game is converted to WSTC). For WSTS games, subtler arguments are needed. This can be seen in the work of Kukushkin (1994), who established the existence of NE in the presence of *additive* aggregation.

The pseudo-potential provides a unified proof of the existence of NE in WSTS/C games with (general) aggregation, assuming only that strategy sets are compact (see Theorems 2, 3, and Section 5). It also helps to establish the stability of NE. If the games have finite strategy sets then, for generic payoffs, sequential best-replies converge to NE (see Remark 1). This yields a partial generalization of the results in Kukushkin (2004), who obtained convergence for all payoffs but with a considerably stronger notion of strategic substitutes.[7] Remark 1, furthermore, clarifies the relation between our findings and those of Dindoš and Mezzetti (2003). And, if strategy sets are convex and best replies are unique, the limit points of certain adaptive processes with simultaneous best replies—reminiscent of fictitious play—are also NE, as follows from the results of Huang (2002) (see Remark 2).

Since the existence of NE in our WSTS/C games (and indeed in all pseudo-potential games) relies only on the compactness of strategy sets, we are able to incorporate nonconvexities that are bound to arise when indivisibilities are present in the underlying economic model. To highlight this point, we re-examine Cournot oligopoly in the quite general setting of Amir (1996) (or, alternatively, Novshek, 1985). But we extend their models by dropping the hypothesis (maintained by both) that firms' strategy sets are convex. It turns out that we get a WSTS game with aggregation,[8] and hence NE exist, generalizing the results of Amir and Novshek. In particular, NE exist in the "discrete" Cournot model, where each firm can produce only finitely many levels of output (a fact already observed by Shapley (1994) when demand and costs are linear).

The paper is organized as follows. In Section 2 we introduce the notion of WSTS/C games and, for ease of exposition, we start (as in Kukushkin, 1994) with simple additive aggregation. Pseudo-potential games are introduced in Section 3. It is shown in Theorem 1 that these games include WSTS and WSTC games. They always possess an NE (Propo-

[5] Throughout, we confine ourselves to pure strategies; so NE will always mean "pure-strategy NE."

[6] Supermodular games of Milgrom and Roberts (1990) are basically the STC that we mentioned in the beginning: marginal returns to increasing one's strategy rise with increases in competitors' strategies. However, strategy sets in their setup do not have to be totally ordered and are only required to be complete lattices.

[7] The notion of (strict) strategic substitutes in Kukushkin (2004) is another ordinal version of STS, defined by means of the "dual strong single crossing property" that we recall in Section 2.

[8] While this fact is frequently alluded to, we did not find an explicit derivation of it outside the setting of convex strategy sets. For the sake of completeness, we establish it in Section 7 for compact strategy sets.

sition 1 and Theorem 2 in Section 4). The convergence of certain adaptive processes to NE is mentioned in Remarks 1 and 2. We develop the concept of general aggregation in Section 5, and verify that our results remain intact. In Section 6, we extend our approach to include discontinuous reaction functions (see Theorem 3 and Corollary 1). Finally, in Section 7, we use this extension to show that indivisibilities in production do not disturb the existence of NE in the Cournot oligopoly model (see Theorem 4).

2. Weak strategic substitutes and complements with aggregation

Consider a set of players $N = \{1, 2, \ldots, n\}$. Each $i \in N$ has a set of strategies S^i, which is a nonempty compact subset of R_+. Put $S \equiv S^1 \times \cdots \times S^n$. For any $s = (s^1, \ldots, s^n) \in S$ and $t \in S^i$, denote $(s^1, \ldots, s^{i-1}, t, s^{i+1}, \ldots, s^n)$ by $(s \,|_i\, t)$; $(s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^n)$ by s_{-i}; and $\sum_{j \in N\setminus\{i\}} s^j$ by $\bar{s}_{-i}$. The payoff function $\pi^i : S \to R$ of player i depends only upon his own strategy s^i and the additive aggregate[9] $\bar{s}_{-i}$ of others' strategies. So, with a slight abuse of notation, we will write $\pi^i(s^i, \bar{s}_{-i})$ for $\pi^i(s)$, and view π^i as defined on the domain $S^i \times \overline{S}_{-i}$, where $\overline{S}_{-i} \equiv \sum_{j \in N\setminus\{i\}} S^j$.

For any choice $s_{-i} \in \prod_{j \in N\setminus\{i\}} S^j$ of others' strategies, the set $\beta^i(\bar{s}_{-i})$ of *best replies* of player i is given by

$$\beta^i(\bar{s}_{-i}) = \arg\max_{t \in S^i} \pi^i(t, \bar{s}_{-i}).$$

We assume it to be always nonempty (as will follow if $\pi^i(t, \bar{s}_{-i})$ is continuous in t for every $\bar{s}_{-i}$). Recall that $s = (s^1, \ldots, s^n) \in S$ is a *Nash equilibrium* (NE) if

$$s^i \in \beta^i(\bar{s}_{-i})$$

for all $i \in N$.

Finally, let us recall the notion of *strategic substitutes*. This was introduced in Bulow et al. (1985) (see also Fudenberg and Tirole, 1986; Tirole, 1988). Its more general, ordinal version[10] refers to games satisfying the *dual strong single crossing property* (DSSCP):

$$\pi^i\big(s^i, \bar{t}_{-i}\big) \leqslant \pi^i\big(t^i, \bar{t}_{-i}\big) \quad \Rightarrow \quad \pi^i\big(s^i, \bar{s}_{-i}\big) < \pi^i\big(t^i, \bar{s}_{-i}\big), \tag{1}$$

for every $i \in N$ and s, $t \in S$ with $s^i > t^i$ and $\bar{s}_{-i} > \bar{t}_{-i}$. It is then evident that every selection $b^i : \overline{S}_{-i} \to S^i$ from β^i is a nonincreasing function of $\bar{s}_{-i}$.[11]

In Kukushkin (2004), games with DSSCP are referred to as "games with strict strategic substitutes." The fact that these games have downward sloping reaction functions indeed suggests a broader (weaker) definition of games of strategic substitutes. When there is multiplicity of best replies, two distinct options come naturally to mind. One could require that $\sup \beta^i(x_1) \leqslant \inf \beta^i(x_2)$ whenever $x_1 > x_2$; in other words, that *every* selection from

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9 For a more general notion of aggregation see Section 5.

10 See, e.g., Amir (1996) and Kukushkin (2004).

11 In fact, this is true even when both inequalities in (1) are weak, provided the best reply correspondence of each player is single-valued. This version of (1) is called the *dual single crossing property* (DSCP), and taken to be the definition of strategic substitutes in Kukushkin (2004).

β^i be nonincreasing. The other option (that we take in this paper, following Kukushkin, 1994) is to require only that there be *some* selection b^i from β^i with this property. Unlike Kukushkin (1994), we do not require upper hemi-continuity of β^i, but instead suppose that it admits a continuous[12] selection b^i.

Formally, we say that $\Gamma = (N, S^1, \ldots, S^n, \pi^1, \ldots, \pi^n)$ is a game of *weak strategic substitutes* (WSTS) with aggregation if, for every $i \in N$, there exists a function $b^i : \overline{S}_{-i} \to S^i$ such that:

(i) $b^i(x) \in \beta^i(x)$ for all $x \in \overline{S}_{-i}$,
(ii) b^i is continuous[13] on $\overline{S}_{-i}$, and
(iii) $b^i(x) \leqslant b^i(y)$ whenever $x > y$.

A game of *weak strategic complements* (WSTC) with aggregation is defined exactly as above, except for replacing "$x > y$" by "$x < y$" in (iii).

3. Pseudo-potential games

Consider a game $\widetilde{\Gamma} = (N, S^1, \ldots, S^n, \tilde{\pi}^1, \ldots, \tilde{\pi}^n)$ in which the players and their strategy-sets are as before, but payoff functions $\tilde{\pi}^i : S \to R$ are allowed to take a general form (i.e., they need not depend on competitors' strategies only via the aggregate). We say that $\widetilde{\Gamma}$ is a *pseudo-potential game* if there exists a continuous function $P : S \to R$ such that, for all $i \in N$ and all $s \in S$,

$$\arg\max_{t \in S^i} \tilde{\pi}^i(s \mid_i t) \supset \arg\max_{t \in S^i} P(s \mid_i t). \tag{2}$$

In other words, each player's best reply correspondence in the game $\Gamma^* = (N, S^1, \ldots, S^n, P, \ldots, P)$ is included in that of $\widetilde{\Gamma}$: it suffices for a player to maximize the pseudo-potential P, rather than his real payoff $\tilde{\pi}^i$, in order to get to *some* best reply. One may therefore think of the pseudo-potential P as a convenient common proxy for all the different payoff functions $\tilde{\pi}^i$, $i \in N$, in the analysis of NE of $\widetilde{\Gamma}$. (For, as is evident, NE of Γ^* are a fortiori NE of $\widetilde{\Gamma}$.)

Our main result is:

Theorem 1. *A game of weak strategic substitutes or complements with aggregation is a pseudo-potential game.*

Proof. We start by considering WSTS games with aggregation. Let $\Gamma = (N, S^1, \ldots, S^n, \pi^1, \ldots, \pi^n)$ be such a game, and, for all $i \in N$, let $b^i : \overline{S}_{-i} \to S^i$ be a continuous and nonincreasing best-reply selection.

[12] The case of discontinuous selections will be considered later, in Section 6.

[13] This requirement holds vacuously if the strategy sets S^i are finite.

Denote by Σ_{-i} the convex hull of $\bar{S}_{-i}$, which is obviously compact. We extend b^i, in a piecewise-linear fashion, to a function τ^i, defined on the entire domain Σ_{-i}. Thus τ^i coincides with b^i on $\bar{S}_{-i}$; and, if $z \in \Sigma_{-i} \backslash \bar{S}_{-i}$ we define

$$\tau^i(z) = \frac{y-z}{y-x} b^i(x) + \frac{z-x}{y-x} b^i(y),$$

where $x = \max\{w \in \bar{S}_{-i} \mid w \leqslant z\}$ and $y = \min\{w \in \bar{S}_{-i} \mid w \geqslant z\}$. (Notice that $z \in (x, y) \subset \Sigma_{-i} \backslash \bar{S}_{-i}$ in this case.) Furthermore, we enhance the domain of τ^i to include the intervals $[-1, \min \Sigma_{-i}]$ and $[\max \Sigma_{-i}, \max \Sigma_{-i} + 1]$, by setting: $\tau^i(-1) = \max S^i$, $\tau^i(\max \Sigma_{-i} + 1) = 0$. We then extend τ^i linearly on $(-1, \min \Sigma_{-i})$ and $(\max \Sigma_{-i}, \max \Sigma_{-i} + 1)$.

Notice that τ^i inherits continuity, and the property of being nonincreasing, from b^i. For every $i \in N$, now define $F_i : S^i \to R$ by

$$F_i(s^i) = \int_{-1}^{\max(\Sigma_{-i})+1} \min(\tau^i(x), s^i)\,\mathrm{d}x.$$

Consider the continuous[14] function $P : S^1 \times \cdots \times S^n \to R$ given by[15]

$$P(s^1, \ldots, s^n) = -\sum_i s^i - \sum_{i<j} s^i s^j + \sum_i F_i(s^i). \tag{3}$$

We claim that P renders Γ into a pseudo-potential game.[16] To check this, fix $s \in S$. Note that for any $t \in S^i$

$$P(s \mid_i t) = \left[-t(\bar{s}_{-i} + 1) + F_i(t)\right] + \Bigg[-\sum_{j \neq i} s^j - \sum_{\substack{j<k \\ j,k \neq i}} s^j s^k + \sum_{j \neq i} F_j(s^j)\Bigg]. \tag{4}$$

It is clear that t maximizes $P(s \mid_i t)$ if and only if it maximizes the first (bracketed) term in (4) (for the given s_{-i}). We will deduce from this that

$$\arg\max_{t \in S^i} P(s \mid_i t) = \{\tau^i(\bar{s}_{-i})\}\left(= \{b^i(\bar{s}_{-i})\}\right). \tag{5}$$

For every $t < \tau^i(\bar{s}_{-i})$, note that

$$-t(\bar{s}_{-i} + 1) + F_i(t) = \int_{-1}^{\max(\Sigma_{-i})+1} \min(\tau^i(x), t)\,\mathrm{d}x - \int_{-1}^{\bar{s}_{-i}} t\,\mathrm{d}x \tag{6}$$

14 Note that when strategy sets S^i are convex, the potential P is multi-concave (this is the property that is needed in Remark 2). Indeed, it is clear from (4) that multi-concavity of P is tantamount to concavity of the functions F_i. This, in turn, follows from the fact that each τ^i is nonincreasing.

15 A function of a similar form first came to our attention in Huang (2002). He, however, defined it under more restrictive assumptions on best replies, in the context of certain Cournot oligopoly games with convex strategy sets, in order to study properties of fictitious play.

16 For a geometric intuition of this proof, see the Appendix in Dubey et al. (2004), the discussion paper on which this article is based.

$$= \int_{-1}^{\max(\Sigma_{-i})+1} \min\bigl(\tau^i(x), t\bigr)\,\mathrm{d}x - \int_{-1}^{\bar{s}_{-i}} \min\bigl(\tau^i(x), t\bigr)\,\mathrm{d}x, \tag{7}$$

since $t < \tau^i(\bar{s}_{-i})$, and since $\tau^i(x) \geqslant \tau^i(\bar{s}_{-i})$ for $x \in [-1, \bar{s}_{-i}]$ by the monotonicity of τ^i. The term displayed in (7) obviously equals

$$\int_{\bar{s}_{-i}}^{\max(\Sigma_{-i})+1} \min\bigl(\tau^i(x), t\bigr)\,\mathrm{d}x. \tag{8}$$

However,

$$\int_{\bar{s}_{-i}}^{\max(\Sigma_{-i})+1} \min\bigl(\tau^i(x), t\bigr)\,\mathrm{d}x < \int_{\bar{s}_{-i}}^{\max(\Sigma_{-i})+1} \min\bigl(\tau^i(x), \tau^i(\bar{s}_{-i})\bigr)\,\mathrm{d}x, \tag{9}$$

since $\min(\tau^i(x), t) \leqslant \min(\tau^i(x), \tau^i(\bar{s}_{-i}))$, and since the inequality is strict for all $x \in [\bar{s}_{-i}, \max(\Sigma_{-i}) + 1]$ which are sufficiently close to $\bar{s}_{-i}$ (on account of the assumption that $t < \tau^i(\bar{s}_{-i})$, and the continuity of τ^i). Just as in (6)–(8), it can be seen that

$$-\tau^i(\bar{s}_{-i})(\bar{s}_{-i} + 1) + F_i\bigl(\tau^i(\bar{s}_{-i})\bigr) = \int_{\bar{s}_{-i}}^{\max(\Sigma_{-i})+1} \min\bigl(\tau^i(x), \tau^i(\bar{s}_{-i})\bigr)\,\mathrm{d}x,$$

and (9) now implies that

$$-t(\bar{s}_{-i} + 1) + F_i(t) < -\tau^i(\bar{s}_{-i})(\bar{s}_{-i} + 1) + F_i\bigl(\tau^i(\bar{s}_{-i})\bigr).$$

It follows from (4) that

$$P(s \mid_i t) < P\bigl(s \mid_i \tau^i(\bar{s}_{-i})\bigr) \quad \text{for } t < \tau^i(\bar{s}_{-i}). \tag{10}$$

Now, if $t > \tau^i(\bar{s}_{-i})$, then

$$-t(\bar{s}_{-i} + 1) + F_i(t) = \int_{-1}^{\max(\Sigma_{-i})+1} \min\bigl(\tau^i(x), t\bigr)\,\mathrm{d}x - \int_{-1}^{\bar{s}_{-i}} t\,\mathrm{d}x \tag{11}$$

$$= \int_{\bar{s}_{-i}}^{\max(\Sigma_{-i})+1} \tau^i(x)\,\mathrm{d}x - \int_{-1}^{\bar{s}_{-i}} \bigl(t - \min\bigl(\tau^i(x), t\bigr)\bigr)\,\mathrm{d}x$$

$$< \int_{\bar{s}_{-i}}^{\max(\Sigma_{-i})+1} \tau^i(x)\,\mathrm{d}x. \tag{12}$$

The last inequality follows from the fact that $t - \min(\tau^i(x), t) \geqslant 0$ and the fact that this inequality is strict for all $x \in [-1, \bar{s}_{-i}]$ which are sufficiently close to $\bar{s}_{-i}$ (on account of the assumption that $t > \tau^i(\bar{s}_{-i})$, and the continuity of τ^i). By the monotonicity of τ^i,

$$
\begin{aligned}
&-\tau^i(\bar{s}_{-i})(\bar{s}_{-i}+1)+F_i\big(\tau^i(\bar{s}_{-i})\big)\\
&\quad=\int_{\bar{s}_{-i}}^{\max(\Sigma_{-i})+1}\tau^i(x)\,\mathrm{d}x-\int_{-1}^{\bar{s}_{-i}}\big(\tau^i(\bar{s}_{-i})-\min\big(\tau^i(x),\tau^i(\bar{s}_{-i})\big)\big)\,\mathrm{d}x\\
&\quad=\int_{\bar{s}_{-i}}^{\max(\Sigma_{-i})+1}\tau^i(x)\,\mathrm{d}x,
\end{aligned}
$$

and thus (11) and (12) imply that

$$-t(\bar{s}_{-i}+1)+F_i(t)<-\tau^i(\bar{s}_{-i})(\bar{s}_{-i}+1)+F_i\big(\tau^i(\bar{s}_{-i})\big).$$

From (4) it follows that

$$P(s\mid_i t)<P\big(s\mid_i \tau^i(\bar{s}_{-i})\big)\quad \text{for } t>\tau^i(\bar{s}_{-i}).$$

Combining this with (10) leads to (5).

Equality (5) shows that b^i is the best-reply (single-valued) correspondence of i in the game $(N, S^1, \ldots, S^n, P, \ldots, P)$. Since b^i, to begin with, was a selection from the best reply correspondence of $\Gamma = (N, S^1, \ldots, S^n, \pi^1, \ldots, \pi^n)$, we conclude that Γ is a pseudo-potential game. This proves the theorem for WSTS games.

When Γ is a WSTC game with aggregation, it can be converted into a WSTS game with aggregation by an appropriate change of its (additive) aggregator. For details, see the Section 5 where the concept of general aggregation is developed, and then see Remark 4 where this change is described. □

4. NE in pseudo-potential games

The existence of a pseudo-potential in a game has some important ramifications. It is easy to establish:

Proposition 1. *A pseudo-potential game has an NE.*

Proof. Let $\widetilde{\Gamma} = (N, S^1, \ldots, S^n, \tilde{\pi}^1, \ldots, \tilde{\pi}^n)$ be a game with pseudo-potential P. Suppose

$$s=\big(s^1,\ldots,s^n\big)\in\arg\max_{(t^1,\ldots,t^n)\in S}P\big(t^1,\ldots,t^n\big)$$

(such an s exists because P is continuous and S is compact). If s is not an NE of $\Gamma^* = (N, S^1, \ldots, S^n, P, \ldots, P)$, then $P(s\mid_i t) > P(s)$ for some $t \in S^i$, contradicting that s maximizes P.

But any NE of Γ^* is a fortiori an NE of $\widetilde{\Gamma}$, since best replies in Γ^* are by definition best replies in $\widetilde{\Gamma}$. □

In conjunction with Theorem 1, this immediately yields:

Theorem 2. *Any WSTS or WSTC game with aggregation has an NE.*

For supermodular WSTC games (even without aggregation), Theorem 2 was established in Milgrom and Roberts (1990). For WSTS games with additive aggregation, Theorem 2 follows from Kukushkin (1994), provided best reply correspondences are upper hemi-continuous.

We shall show, however, in Section 5 that Theorem 2 remains intact even when the concept of aggregation is generalized, and no longer restricted to being additive. In that setting, Theorem 2 shows the existence of NE in games that lie well beyond the domain considered in Milgrom and Roberts (1990) and Kukushkin (1994). Furthermore, the general aggregators that we admit may fail to be monotonic in the strategies themselves. Thus our WSTS/C games with general aggregation need not even be WSTS/C in the ordinary sense: the best reply selection b^i of player i can be non-monotonic with respect to others' strategies.

Existence of a pseudo-potential in a game implies certain stability properties of the set of NE. We summarize them in the following remarks.

Remark 1 (*Generic convergence of sequential best replies with finite strategy sets*). Given finite strategy sets, WSTS/C games have single-valued best reply correspondences for generic payoffs. In this situation the pseudo-potential is an (ordinal) best-reply potential[17] as defined in Voorneveld (2000), i.e., equality holds in place of inclusion in (2). It is then straightforward to check that there are no best-response cycles in the game; i.e., if players start with an arbitrary strategy profile, and each player (one at a time) unilaterally deviates to his unique best reply (if he does not happen to be there already), then no cycles can occur along the way and the process terminates in an NE after finitely many steps.

Thus, Theorem 1 provides a partial generalization of a "no-cycling" result of Kukushkin (2004), by admitting a more general notion of strategic substitutes. (As was said, Kukushkin, 2004 defined strict STS by means of DSSCP of the payoff functions, and it immediately implies WSTS in our sense.)

Dindoš and Mezzetti (2003) also analyze convergence of better-reply dynamics, in which the turn of a player to unilaterally deviate, as well as the payoff-improving strategy to which he deviates, are determined randomly. They obtain convergence results for games of local[18] strategic substitutes or complements with aggregation, but in the setup where players have infinite and *convex* strategy sets, and quasi-concave payoffs. (With finite strategy sets, their stochastic better-reply dynamics leads to convergence to an NE in a large class of games with aggregation, but in order for this to be the case without the stochasticity assumption, it becomes necessary to postulate STS/C).

Remark 2 (*Convergence of simultaneous best replies with convex strategy sets*). Consider a WSTS/C game with aggregation and with convex strategy sets, and assume that all best reply correspondences are single-valued.[19] Again, the pseudo-potential is a best-

[17] See Morris and Ui (2004) for more discussion of this notion.

[18] In other words, best reply correspondences must be single-valued and differentiable, and their derivatives must have the same sign in open neighborhoods of NE.

[19] In the "standard" case, when the payoff of each player is strictly concave with respect to his own unilateral deviations, the best reply will be unique.

reply potential in this situation. Suppose that in every period players simultaneously choose the best reply to their current *conjectures* of others' strategies. Formally, let $\{\lambda_t\}_{t=1}^{\infty}$ be a sequence of positive numbers such that[20] $\sum_{t=1}^{\infty} \lambda_t = \infty$ and $\sum_{t=1}^{\infty} \lambda_t^2 < \infty$. Let $\{s(t)\}_{t=1}^{\infty} = \{(s^1(t), \ldots, s^n(t))\}_{t=1}^{\infty}$ be a sequence of strategy-profiles with the following property. At every period t, $s^i(t)$ is the unique best reply of i to his conjecture $\sigma_{-i}(t)$ of the true strategies of other players. Conjectures are defined recursively: $\sigma_{-i}(1)$ are arbitrary, and for all $j \in N\backslash\{i\}$

$$\sigma^j(t) = \lambda_t s^j(t-1) + (1-\lambda_t)\sigma^j(t-1).$$

(Setting $\lambda_t = 1/t$ is reminiscent of fictitious play.)

It was established by Huang (2002, Proposition 3.3.1) that all limit points of the sequence $\{s(t)\}_{t=1}^{\infty}$ are NE for best-reply potential games, provided the potential is continuous and "multi-concave" (i.e., concave separately in each player's unilateral deviations). But the (continuous) pseudo-potential P, constructed in the proof of Theorem 1, is easily seen to be multi-concave (see footnote 14 in that proof for WSTS games, and Remark 4 for WSTC games).

5. Non-additive aggregation

Adding up players' strategies is but one way of aggregating them. However, there are also many kinds of strategic interaction which, at first glance, look alien to our framework. It is only when appropriate aggregators $\alpha : S_{-i} \to R$ are constructed for them, that their hidden structure is unmasked and they fit into our framework, with $\bar{s}_{-i}$ replaced by $\alpha(s_{-i})$ and $\overline{S}_{-i}$ by $\alpha(S_{-i})$.

Before giving examples, let us define our class of aggregators. Denote

$$s_{-i}^*(k) \equiv \sum_{\substack{i_1<i_2<\cdots<i_k \\ i_1,i_2,\ldots,i_k \neq i}} s_{i_1} \cdots s_{i_k}$$

for $1 \leqslant k \leqslant n-1$ (i.e., $s_{-i}^*(k)$ is the sum of all possible products of k distinct strategies picked from $s_{-i} = (s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^n)$; and so for $k=1$ we get $s_{-i}^*(1) = \bar{s}_{-i}$.) Let $a_1, \ldots, a_{n-1}$ be scalars, and define

$$\alpha(s_{-i}) \equiv \sum_{k=1}^{n-1} a_k s_{-i}^*(k).$$

For the moment assume, by way of simplicity, that the scalars a_k are such that $\alpha(s_{-i}) \geqslant 0$ for all $i \in N$ and all $s_{-i} \in S_{-i}$. (This restriction can be dropped, see Remark 3.) Notice that the aggregator $\alpha(s_{-i})$ is the *same* linear combination of $\{s_{-i}^*(k)\}_{k=1}^{n-1}$ for all $i \in N$.

Our aggregators are seemingly abstruse. We shall now give four examples to illustrate how they might arise in a natural manner. In the first three examples, each player i chooses

[20] Here λ_t represents the weight given to the most recent observation in period t, and may be interpreted as the "speed" with which players update their conjectures.

effort level $s^i \in [0, B^i]$ to apply to the personal task faced by him. This gives rise to the probability $p_i(s^i)$ of "success" in his task, where $p_i : [0, B^i] \to [0, 1]$ is a continuous and strictly increasing function with $p_i(0) = 0$ and $p_i(B^i) = 1$. By relabeling effort levels if necessary, we take $B^i = 1$ and $p_i(s^i) = s^i$. The events of individual success are assumed to be independent across different players. Furthermore, for ease of calculation, we assume for the moment that there are three players ($N = \{1, 2, 3\}$), and that each $i \in N$ incurs quadratic cost $c_i(s^i)^2$, on account of his effort s^i, for some constant $c_i > 0$.

Example 1 (*Team projects with complementary tasks*). Each player's task is critical to the success of the team's project. Thus $s^1 s^2 s^3$ is the probability that the project will succeed. Suppose $r_i > 0$ is the utility to player i of a successful project. This yields the payoff function

$$\pi^i\big(s^1, s^2, s^3\big) = r_i s^1 s^2 s^3 - c_i\big(s^i\big)^2 = r_i s^i \alpha(s_{-i}) - c_i\big(s^i\big)^2,$$

where $\alpha(s_{-i})$ is the aggregator $s^*_{-i}(2) = s^j s^k$ (and, $N\backslash\{i\} = \{j, k\}$). Then i's best reply is

$$\beta^i\big(\alpha(s_{-i})\big) = \left\{\min\left(\frac{r_i \alpha(s_{-i})}{2c_i}, 1\right)\right\},$$

which is a nondecreasing function of $\alpha(s_{-i})$; and shows that we have a WSTC game with aggregation (when $\bar{s}_{-i}$ is replaced by $\alpha(s_{-i})$).

Example 2 (*Team projects with substitutable tasks*). Here we suppose that each player by himself can make the project successful. Then the probability that the project is successful is

$$\begin{aligned} f\big(s^1, s^2, s^3\big) &= 1 - \big(1 - s^1\big)\big(1 - s^2\big)\big(1 - s^3\big) \\ &= s^1 + s^2 + s^3 - s^1 s^2 - s^1 s^3 - s^2 s^3 + s^1 s^2 s^3, \end{aligned}$$

and the payoff to player i is

$$\pi^i\big(s^1, s^2, s^3\big) = r_i f\big(s^1, s^2, s^3\big) - c_i\big(s^i\big)^2 = r_i s^i\big[1 - \alpha(s_{-i})\big] + \alpha(s_{-i}) - c_i\big(s^i\big)^2,$$

where $\alpha(s_{-i})$ is the aggregator $s^*_{-i}(1) - s^*_{-i}(2)$. Thus

$$\beta^i\big(\alpha(s_{-i})\big) = \left\{\min\left(\frac{r_i[1 - \alpha(s_{-i})]}{2c_i}, 1\right)\right\}$$

is a nonincreasing function of $\alpha(s_{-i})$, and so this example describes a WSTS game with aggregator α.

Example 3 (*Tournaments*). Assume that a reward of r dollars is shared by the group of players who succeed. If only one player succeeds, he gets r for sure; if exactly two succeed, each gets r with probability $1/2$; if all three succeed, each gets r with probability $1/3$. By rescaling utilities, we may assume w.l.o.g. that r dollars yield r utilities to each player. Then the expected value of the reward to i is

$$rs^i(1-s^j)(1-s^k)+\frac{r}{2}s^is^j(1-s^k)+\frac{r}{2}s^is^k(1-s^j)+\frac{r}{3}s^is^js^k$$
$$=rs^i\left[1-\frac{1}{2}s^j-\frac{1}{2}s^k+\frac{1}{3}s^js^k\right]=rs^i\left[1-\alpha(s_{-i})\right],$$

where $\alpha(s_{-i})=\frac{1}{2}s^*_{-i}(1)-\frac{1}{3}s^*_{-i}(2)$. Therefore each player's payoff function is

$$\pi^i(s^1,s^2,s^3)=rs^i\left[1-\alpha(s_{-i})\right]-c_i(s^i)^2.$$

Consequently,

$$\beta^i(\alpha(s_{-i}))=\left\{\min\left(\frac{r[1-\alpha(s_{-i})]}{2c_i},1\right)\right\}$$

is a nonincreasing function of $\alpha(s_{-i})$, and therefore tournaments are also WSTS games with aggregator α.

Example 4 (*Team projects and tournaments with non-convex strategy sets and general cost functions*). Consider strategy sets S^i that are arbitrary closed subsets of $[0,1]$. Let g_i be a continuous cost function (not necessarily quadratic) on $[0,1]$, for each player i. We claim that the WSTS or WSTC character of the games in Examples 1–3 will not be affected by these generalizations. Indeed, it is easy to see that the modified payoff function[21] π^i from Example 1 satisfies the *strong single crossing property* (SSCP) with respect to the aggregator α:

$$\pi^i(s^i,\alpha(t_{-i}))\geqslant\pi^i(t^i,\alpha(t_{-i}))\Rightarrow\pi^i(s^i,\alpha(s_{-i}))>\pi^i(t^i,\alpha(s_{-i})),$$

for every $s,t\in[0,1]^3$ with $s^i>t^i$ and $\alpha(s_{-i})>\alpha(t_{-i})$. Thus, a fortiori, the SSCP continues to hold when the strategy sets are closed *subsets* of $[0,1]$. It is then evident (alternatively, see Theorem 4 of Milgrom and Shannon, 1994) that every selection from β^i is a nondecreasing function of $\alpha(s_{-i})$. Similarly, the modified payoff functions π^i from Examples 2 and 3 satisfy the DSSCP with respect to α:

$$\pi^i(s^i,\alpha(t_{-i}))\leqslant\pi^i(t^i,\alpha(t_{-i}))\Rightarrow\pi^i(s^i,\alpha(s_{-i}))<\pi^i(t^i,\alpha(s_{-i})),$$

for every $s,t\in[0,1]^3$ with $s^i>t^i$ and $\alpha(s_{-i})>\alpha(t_{-i})$. Again, the DSSCP continues to hold when the strategy sets are closed subsets of $[0,1]$, and every selection from β^i is a nonincreasing function of $\alpha(s_{-i})$.

As we said, our results remain intact if we postulate that the payoff to any player i depends only upon his own strategy s^i and the aggregate $\alpha(s_{-i})$ of others' strategies. Other than the obvious change of notation ($\bar{s}_{-i}$ replaced by $\alpha(s_{-i})$ and $\overline{S}_{-i}$ by $\alpha(S_{-i})$), the only variation needed is in the proof of Theorem 1. Given a WSTS game with aggregator α, where $\alpha(s_{-i})=\sum_{k=1}^{n-1}a_ks^*_{-i}(k)$, we redefine P (which was defined for the additive aggregator in (3)) as follows:

$$P(s^1,\ldots,s^n)=-\sum_i s^i-\sum_{k=1}^{n-1}a_k\cdot\sum_{i_1<i_2<\cdots<i_{k+1}}s^{i_1}\cdots s^{i_{k+1}}+\sum_i F_i(s^i). \tag{13}$$

[21] Obtained by replacing $c_i(\cdot)^2$ by $g_i(\cdot)$ throughout.

Note that for any $i \in N$,

$$
\begin{aligned}
P(s\,|_i\,t) &= \left[-t\left(\sum_{k=1}^{n-1} a_k s^*_{-i}(k) + 1\right) + F_i(t)\right] \\
&\quad + \left[-\sum_{j\neq i} s^j - \sum_{k=1}^{n-2} a_k s^*_{-i}(k+1) + \sum_{j\neq i} F_j\big(s^j\big)\right] \qquad (14) \\
&= \big[-t\big(\alpha(s_{-i}) + 1\big) + F_i(t)\big] \\
&\quad + \left[-\sum_{j\neq i} s^j - \sum_{k=1}^{n-2} a_k s^*_{-i}(k+1) + \sum_{j\neq i} F_j\big(s^j\big)\right]. \qquad (15)
\end{aligned}
$$

The above equality replaces (4) in the proof of Theorem 1, and the rest of the arguments hold exactly as before.

It must be mentioned (as was pointed out to us by an anonymous referee) that Examples 1–4 above admit *weighted potentials*[22] (defined in Monderer and Shapley, 1996), and can therefore be analyzed without recourse to our Theorem 1 (for non-additive aggregation). Our approach does provide an alternative view, which we hope is not without some utility. And we conjecture that there are games of our variety which are not weighted potential games.

Remark 3 (*Aggregation without the positivity requirement*). The requirement that aggregation be nonnegative can be dropped. If $\alpha(s_{-i})$ is negative for some $s_{-i} \in S_{-i}$, we can define another aggregator $\tilde{\alpha}$ by $\tilde{\alpha}(s_{-i}) \equiv \alpha(s_{-i}) + a$ for all $i \in N$ and all $s_{-i} \in S_{-i}$. Clearly, for large enough a, $\tilde{\alpha}(s_{-i})$ is always nonnegative. It is obvious that if i's payoff is a function of s_i and $\alpha(s_{-i})$, then it is representable also as a function of s_i and $\tilde{\alpha}(s_{-i})$, and any nonincreasing and continuous best-reply selection remains such after this change of variables. Our analysis holds with these "non-homogeneous"aggregations just as well. One only has to add $s^*_{-i}(0) \equiv 1$ to the set $\{s^*_{-i}(k)\}_{k=1}^{n-1}$, and allow aggregators $\alpha(s_{-i})$ to be linear combinations of $\{s^*_{-i}(k)\}_{k=0}^{n-1}$, not just $\{s^*_{-i}(k)\}_{k=1}^{n-1}$.

Remark 4 (*WSTC games with aggregation*). The previous remark also enables us to include WSTC games with aggregation in our approach. Indeed, given a WSTC game with aggregator α, the payoff function of each player i can be obviously redefined to depend on s^i and $\tilde{\alpha}(s_{-i}) \equiv -\alpha(s_{-i})$, instead of s^i and $\alpha(s_{-i})$. Consequently, if a best-reply selection b^i is a nondecreasing function of α (which is the case for WSTC), it turns into $\tilde{b}^i(\tilde{\alpha}) \equiv b^i(-\tilde{\alpha})$, a nonincreasing function of $\tilde{\alpha}$, and our analysis goes through by Remark 3. Note that this trick is purely technical, and does not change the real WSTC character of the game, if the aggregator α is increasing in s_{-i}: while $\tilde{b}^i(\tilde{\alpha}(s_{-i}))$ is a nonincreasing function of $\tilde{\alpha}(s_{-i})$, it remains nondecreasing in the underlying basic variable s_{-i}.

[22] In other words, the change in any player's payoff from switching between any two of his strategies (holding other players' strategies fixed) is proportional to the change in the potential function.

Assume now that all strategy-sets are convex. Using the same argument as in footnote 14 and equalities (14) and (15), it can be shown that the pseudo-potential P (when the aggregator used is $\tilde{\alpha}(s_{-i}) + a$, for sufficiently large a) is multi-concave.

6. Discontinuous best reply selections

We do not know if continuity of our best reply selections is necessary for the validity of Theorem 1. However, the function P that we constructed in its proof can still be of value, even though P might not be a faithful proxy for discontinuous best replies. We exemplify its use in the proof of the following extension of Theorem 2 on the existence of NE. For simplicity, the result is stated only for WSTS games.

Theorem 3. *The conclusion of Theorem* 2 *remains intact, even without requiring continuity of the best reply selections, provided one of the following assumptions is made*:

- (i) *For every $i \in N$ there exists a best reply selection b^i which is a strictly decreasing function*[23] *of $\bar{s}_{-i}$;*
- (ii) *for every $i \in N$ there exists a best reply selection b^i which is nonincreasing and right-continuous in $\bar{s}_{-i}$;*
- (iii) *for every $i \in N$ there exists a best reply selection b^i which is nonincreasing and left-continuous in $\bar{s}_{-i}$.*

Proof. Given a nonincreasing best reply selection b^i for each player i, construct τ^i and P exactly as in the proof of Theorem 1. P is continuous as before, but this time

$$\arg\max_{t \in S^i} P(s \,|_i\, t) = \Big[\lim_{x \downarrow \bar{s}_{-i}} \tau^i(x), \lim_{x \uparrow \bar{s}_{-i}} \tau^i(x)\Big] \cap S^i \tag{16}$$

for all $i \in N$ and all $s \in S$. In particular, $\arg\max_{t \in S^i} P(s \,|_i\, t)$ need not be single-valued, if τ^i is discontinuous at $\bar{s}_{-i}$. (This is why P may fail to be a pseudo-potential function for the given game.) However, it follows from (16) that

$$b^i(\bar{s}_{-i}) \in \arg\max_{t \in S^i} P(s \,|_i\, t), \tag{17}$$

as before.

Now consider some

$$s = \big(s^1, \ldots, s^n\big) \in \arg\max_{(t^1, \ldots, t^n) \in S} P\big(t^1, \ldots, t^n\big).$$

Suppose first that assumption (i) is satisfied, and let b^i be a strictly decreasing best reply selection for each i. Note that if there is $i \in N$ (say, $i = 1$) such that $s^1 \neq b^1(\bar{s}_{-1})$, then by (17),

$$s' \equiv \big(b^1(\bar{s}_{-1}), s^2, \ldots, s^n\big) \in \arg\max_{(t^1, \ldots, t^n) \in S} P\big(t^1, \ldots, t^n\big).$$

---

[23] For ease of notation, we revert from $\alpha(s_{-i})$ to $\bar{s}_{-i}$ (though the argument holds replacing $\bar{s}_{-i}$ by $\alpha(s_{-i})$ throughout, provided we assume that $\alpha(s_{-i})$ is *strictly* increasing in s_{-i}).

Since s and s' maximize P, $s^i \in \arg\max_{t \in S^i} P(s\,|_i\, t)$ and $(s')^i \in \arg\max_{t \in S^i} P(s'\,|_i\, t)$ for all $i \in N$. Then (16) implies

$$s^2 \in \Big[\lim_{x \downarrow \bar{s}_{-2}} \tau^2(x), \lim_{x \uparrow \bar{s}_{-2}} \tau^2(x)\Big] \cap \Big[\lim_{x \downarrow \overline{s'}_{-2}} \tau^2(x), \lim_{x \uparrow \overline{s'}_{-2}} \tau^2(x)\Big].$$

But clearly $\bar{s}_{-2} \neq \overline{s'}_{-2}$, and so, from the fact that τ^2 is strictly decreasing, the intersection of the above two intervals must be empty. This is a contradiction, so $s^i = b^i(\bar{s}_{-i})$ for all $i \in N$, and s is an NE of Γ.

Next suppose that assumption (ii) holds, and let b^i be a nonincreasing and right-continuous best reply selection for each i. Then, since every τ^i is also nonincreasing and right-continuous, it follows from (16) that

$$b^i(\bar{s}_{-i}) = \min\Big[\arg\max_{t \in S^i} P(s\,|_i\, t)\Big] \tag{18}$$

for all $i \in N$. If (say) $s^1 \notin \arg\max_{t \in S^1} \pi^1(t, \bar{s}_{-1})$, then $b^1(\bar{s}_{-1}) < s^1$ by (18) and the fact that $s^1 \in \arg\max_{t \in S^1} P(s\,|_1\, t)$. Thus for all $j \neq 1$

$$\overline{s'}_{-j} < \bar{s}_{-j}, \tag{19}$$

where (recall)

$$s' \equiv \big(b^1(\bar{s}_{-1}), s^2, \ldots, s^n\big) \in \arg \max_{(t^1, \ldots, t^n) \in S} P\big(t^1, \ldots, t^n\big).$$

Since

$$s^j \in \arg\max_{t \in S^j} P(s\,|_j\, t) \cap \arg\max_{t \in S^j} P(s'\,|_j\, t)$$

for all $j \neq 1$, the conjunction of (16), (19), and the fact that τ^j is nonincreasing, yields

$$s^j = \min\Big[\arg\max_{t \in S^j} P(s'\,|_j\, t)\Big]. \tag{20}$$

The right-hand side of (20) is equal to $b^j(\overline{s'}_{-j})$ by (18). Thus $(s')^j = s^j = b^j(\overline{s'}_{-j})$ for all $j \neq 1$. Since $(s')^1 = b^1(\bar{s}_{-1}) = b^1(\overline{s'}_{-1})$ by definition, s' is an NE of Γ.

Finally, when assumption (iii) holds, the analysis is similar to that for assumption (ii). $\square$

The following result on the existence of NE is a corollary of Theorem 3 (and is also immediately implied by Kukushkin, 1994). It will be used in the next section, when we focus on Cournot oligopoly with indivisibilities in production.

Corollary 1. *Assume that the best-reply correspondence β^i of every player i in Γ is*:

(a) *nonempty-valued*;
(b) *upper hemi-continuous* (*that is, if* $\{(x_n, y_n)\}_{n=1}^{\infty} \subset \overline{S}_{-i} \times S^i$ *is such that* $y_n \in \beta^i(x_n)$ *and* $\lim_{n \to \infty}(x_n, y_n) = (x, y)$, *then* $y \in \beta^i(x)$); *and*
(c) *nonincreasing in the sense that*[24] $\max \beta^i(x_1) \leqslant \min \beta^i(x_2)$ *whenever* $x_1 > x_2$.

Then Γ has an NE.

[24] Note that $\max \beta^i(x)$ and $\min \beta^i(y)$ are well defined by upper hemi-continuity.

Proof. Note that

$$b_r^i(x) = \min \beta^i(x) \quad \text{for all } x \in \overline{S}_{-i}$$

defines a nonincreasing best-reply selection which is right-continuous, and

$$b_l^i(x) = \max \beta^i(x) \quad \text{for all } x \in \overline{S}_{-i}$$

defines a nonincreasing best-reply selection which is left-continuous. Thus, Γ satisfies both (ii) and (iii) of Theorem 3. □

7. Cournot oligopoly with indivisibilities

Consider Cournot oligopoly with the set $N = \{1, 2, \ldots, n\}$ of firms. Following Amir (1996), we assume that the inverse demand function Q is strictly decreasing and log-concave; the cost function c_i of each firm i is strictly increasing and left-continuous; and each firm's monopoly profit (i.e., $xQ(x) - c_i(x)$) becomes negative for large enough x. However, unlike Amir, we allow for indivisibilities in production. The strategy sets S^i, consisting of all possible levels of output producible by firm i, are not required to be convex but just closed.[25] Thus, in particular, the "discrete" Cournot model, in which each firm's outputs consist of indivisible units, will be included in our analysis.

Theorem 4 below extends Theorem 3.1 of Amir (1996), which took $S^i = R_+$. Its proof simply exploits the fact that Cournot oligopoly is a game of strategic substitutes (in the sense expressed in (c) of Corollary 1).

Theorem 4. *Under the above assumptions, Cournot oligopoly has an NE.*

Proof. Amir (1996) showed that the payoff function π^i of each firm i has the DSSCP:

$$\pi^i(x_1, y_2) \leqslant \pi^i(x_2, y_2) \quad \Rightarrow \quad \pi^i(x_1, y_1) < \pi^i(x_2, y_1),$$

where[26] $\pi^i(x, y) = xQ(x + y) - c_i(x)$, for *all* $x_1 > x_2 \geqslant 0$ and $y_1 > y_2 \geqslant 0$. As in Example 4, DSSCP continues to hold when the strategy sets are closed *subsets* of R_+, and thus each β^i is nonincreasing. Also, since profits become negative for large outputs, we may w.l.o.g. restrict the strategy set S^i of each firm i to be compact. Finally, notice that $\beta^i(y) \equiv \arg\max_{x \in S^i} \pi^i(x, y)$ is nonempty-valued and upper hemi-continuous in $y \in \overline{S}_{-i}$, since π^i is continuous in y and upper semi–continuous in (x, y). Thus, according to Corollary 1, NE exists. □

Remark 5 (*Novshek's existence theorem for Cournot oligopoly*). Novshek (1985) established the existence of NE provided Q is strictly decreasing and twice continuously differentiable, and satisfies

$$Q'(x) + xQ''(x) \leqslant 0, \tag{21}$$

[25] We now drop the compactness (i.e., boundedness) requirement.

[26] $x \equiv$output of firm i, $y \equiv$aggregate output of all firms other than i.

for all x below the point where the inverse demand reaches zero. (The cost functions were required to be only weakly, not strictly, increasing). But, as was shown in the proof of Theorem 3 in Novshek (1985), each β^i is still nonempty, upper hemi-continuous and non-increasing. Thus, his existence result is also implied by our Corollary 1.

To extend Novshek's model to allow for indivisibilities, we assume (21) with strict inequality. As in Novshek (1985), this implies that $\frac{\partial}{\partial y}\frac{\partial}{\partial x}[xQ(x+y)] < 0$, i.e., the marginal revenue of any firm is decreasing in the aggregate output of other firms. Therefore

$$x_1 Q(x_1 + y_2) - x_2 Q(x_2 + y_2) > x_1 Q(x_1 + y_1) - x_2 Q(x_2 + y_1)$$

for all $x_1 > x_2 \geqslant 0$ and $y_1 > y_2 \geqslant 0$. This clearly implies the DSSCP of the payoff functions, and so, just as in the previous scenario, our Corollary 1 yields the existence of NE even when the strategy sets S^i are not convex but merely compact.

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References

Amir, R., 1996. Cournot oligopoly and the theory of supermodular games. Games Econ. Behav. 15, 132–148.

Bulow, J.I., Geanakoplos, J.D., Klemperer, P.D., 1985. Multimarket oligopoly: strategic substitutes and complements. J. Polit. Economy 93, 488–511.

Dasgupta, P., Heal, G., 1979. Economic Theory and Exhaustible Resources. Cambridge Univ. Press, Cambridge Economics Handbooks.

Diamond, D.W., Dybvig, P., 1983. Bank runs, deposit insurance, and liquidity. J. Polit. Economy 91, 401–419.

Diamond, P., 1982. Aggregate demand management in search equilibrium. J. Polit. Economy 90, 881–894.

Dindoš, M., Mezzetti, C., 2003. Better-reply dynamic and global convergence to Nash equilibrium in aggregative games. Working paper. University of North Carolina.

Dubey, P., Haimanko, O., Zapechelnyuk, A., 2004. Strategic complements and substitutes, and potential games. Discussion paper # 04-11. Monaster Center for Economic Research, Ben-Gurion University of the Negev, Beer Sheva, Israel.

Fudenberg, D., Tirole, J., 1986. Dynamic Models of Oligopoly. Hardwood Academic, Chur, Switzerland.

Huang, Z., 2002. Fictitious play for games with a continuum of strategies. PhD dissertation. Department of Economics, SUNY at Stony Brook.

Katz, M., Shapiro, C., 1986. Technology adoption in the presence of network externalities. J. Polit. Economy 94, 822–841.

Kukushkin, N.S., 1994. A fixed-point theorem for decreasing mappings. Econ. Letters 46, 23–26.

Kukushkin, N.S., 2004. Best response dynamics in finite games with additive aggregation. Games Econ. Behav. 48, 94–110.

Milgrom, P., Roberts, J., 1990. Rationalizability, learning, and equilibrium in games with strategic complementarities. Econometrica 58, 1255–1277.

Milgrom, P., Shannon, C., 1994. Monotone comparative statics. Econometrica 62, 157–180.

Monderer, D., Shapley, L., 1996. Potential games. Games Econ. Behav. 14, 124–143.

Morris, S., Ui, T., 2004. Best response equivalence. Games Econ. Behav. In press.

Novshek, W., 1985. On the existence of Cournot equilibrium. Rev. Econ. Stud. 52, 85–98.

Shapley, L., 1994. Lecture notes in game theory. Handouts # 9 and 10. Eco 145. Department of Economics, UCLA.

Tirole, J., 1988. The Theory of Industrial Organization. MIT Press, Cambridge, MA.
Vives, X., 1990. Nash equilibrium with strategic complementarities. J. Math. Econ. 19, 305–321.
Voorneveld, M., 2000. Best-response potential games. Econ. Letters 66, 289–295.